

MOMENT ANGLE COMPLEXES AND BIG COHEN-MACAULAYNESS

SHISEN LUO, TOMOO MATSUMURA, AND W. FRANK MOORE

ABSTRACT. Let $\mathcal{Z}_K \subset \mathbb{C}^m$ be the moment angle complex associated to a simplicial complex K on $[m]$, together with the natural action of the torus $T = U(1)^m$. Let $G \subset T$ be a (possibly disconnected) subgroup and $R := T/G$. Let $\mathbb{Z}[K]$ be the Stanley-Reisner ring of K and consider $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z})$ as a subring of $\mathbb{Z}[T^*] := H^*(BT; \mathbb{Z})$. We prove that $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ as a graded module over $\mathbb{Z}[T^*]$. Based on this, we characterize the surjectivity of $\iota^* : H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ (i.e. $H_G^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$) in terms of the vanishing of $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z})$ and discuss its relation to the freeness and the torsion-freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$. For various toric orbifolds X , by which we mean quasi-toric orbifolds or toric Deligne-Mumford stacks, the cohomology of X can be identified with $H_G(\mathcal{Z}_K)$ with appropriate K and G and the above results mean that $H^*(X; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ and that $H^{\text{odd}}(X; \mathbb{Z}) = 0$ if and only if $H^*(X; \mathbb{Z})$ is the quotient $H_R^*(X; \mathbb{Z})$.

1. Introduction

The equivariant cohomology and the ordinary cohomology with \mathbb{Z} -coefficients of a “compact smooth toric space” (including quasi-toric manifolds, complete smooth toric varieties) has been known by the work of Danilov [9], Jurkiewicz [21], and Davis-Januszkiewicz [10]: the equivariant cohomology is the Stanley-Reisner ring of the associated simplicial complex and the ordinary cohomology is the quotient of the equivariant cohomology by linear relations.

The orbifold analogue of these spaces have been also introduced and studied by several people, for example, Lerman-Tolman [23], Borisov-Chen-Smith [4], Poddar-Sarkar [27]. The equivariant cohomology of these toric orbifolds with \mathbb{Z} -coefficients is also known to be the associated Stanley-Reisner rings and the ordinary cohomology is the quotient of the equivariant cohomology over \mathbb{Q} -coefficients, c.f. [9, 21, 4, 27]. However the ordinary cohomology with \mathbb{Z} -coefficients is hard to compute because it is not the quotient of the equivariant cohomology in general, for example, the direct product of weighted projective spaces. The main theme of this paper is to characterize when the ordinary cohomology is the quotient of the equivariant cohomology.

Our approach is to view previously mentioned toric orbifolds as *quotient stacks* given by partial quotients of the moment angle complexes and the *cohomology of stacks* in the sense of [11] (see also [30, 12]). The *moment-angle complex* \mathcal{Z}_K was introduced by Buchstaber and Panov in [7] as a disc-circle decomposition of the Davis-Januszkiewicz universal space associated to a simplicial complex K [10] where they introduced a quasi-toric manifold as a partial quotient of the moment angle manifold associated to a simple polytope.

If K is a simplicial complex on $[m] := \{1, \dots, m\}$, then \mathcal{Z}_K carries a natural action of the torus $T := U(1)^m$. The quotient stack $[\mathcal{Z}_K/G]$ with an appropriate choice of the subgroup $G \subset T$ can be used as a topological model to compute the cohomology of quasi-toric orbifolds [10, 27], symplectic toric orbifolds [23] and toric Deligne-Mumford stacks [4], i.e. *the ordinary cohomology of these toric orbifolds as stacks can be defined as the G -equivariant cohomology* $H_G^*(\mathcal{Z}_K; \mathbb{Z})$. Similarly the equivariant cohomology can be defined as $H_T^*(\mathcal{Z}_K; \mathbb{Z})$ which is isomorphic to the Stanley-Reisner ring $\mathbb{Z}[K]$ as quotient rings of $\mathbb{Z}[T^*] :=$

$H^*(BT; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$. In Section 2, we recall the constructions of those toric orbifolds and the relation to the moment angle complexes to motivate our readers.

In Section 3, we start with proving

Theorem A. *Let $G \subset T$ be a (possibly disconnected) subgroup and $R := T/G$. There is an isomorphism of graded modules over $H^*(BT; \mathbb{Z})$,*

$$H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K]; \mathbb{Z}).$$

Here $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$ is considered as a subring of $\mathbb{Z}[T^*]$ so that u_i 's are linear combinations of x_j 's. It is worth noting that this theorem holds more generally. Namely, the theorem holds for any a topological space X with T -action such that $C^*(ET \times_T X; \mathbb{Z})$ is formal in the category of $H^*(BT; \mathbb{Z})$ -modules up to homotopy in the sense of [14]. See Section 3 for the details.

Based on this isomorphism, we prove our characterization theorem:

Theorem B. *The following are equivalent: (1) $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$; (2) $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to the quotient of $\mathbb{Z}[K]$ by linear terms; (3) $H_G^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$.*

We will explain in Section 4 that, even though $\mathbb{Z}[K]$ might not be finitely generated over $\mathbb{Z}[R^*]$, the vanishing of $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z})$ has the usual meaning in terms of regular sequences, i.e. $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ iff u_1, \dots, u_n form a $\mathbb{Z}[K]$ -regular sequence. Thus we say $\mathbb{Z}[K]$ is big Cohen-Macaulay over $\mathbb{Z}[R^*]$ if (1) is satisfied.

By presenting a toric orbifold X as $[\mathcal{Z}_K/G]$, we obtain the following immediate corollary:

Corollary C. *If X is a toric orbifold stack presented as $[\mathcal{Z}_K/G]$, then*

$$H^*(X; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K]; \mathbb{Z}).$$

Furthermore, $H^*(X; \mathbb{Z})$ is the quotient of Stanley-Reisner ring $\mathbb{Z}[K]$ if and only if one of the following equivalent conditions holds: (i) $H^{\text{odd}}(X; \mathbb{Z}) = 0$; (ii) $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K]; \mathbb{Z}) = 0$.

For example, the cohomology of the weighted projective spaces as stacks are shown to be the quotient of its equivariant cohomology, based on the computation exhibited in [20]. On the other hand, the cohomology of a direct product of weighted projective spaces is not the quotient of its equivariant cohomology. See Section 6 for the details and more examples.

In Section 5, we will discuss the freeness and the torsion-freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$. In particular, we show that the equivariant cohomology of a toric orbifold is torsion-free over $\mathbb{Z}[R^*]$. We also give a certain injectivity theorem of an equivariant cohomology of \mathcal{Z}_K (Theorem 5.17) for a symplectic toric orbifold $[\mathcal{Z}_K/G]$, which give a sufficient condition that $\mathbb{Z}[K]$ is free over a subring of $\mathbb{Z}[R^*]$.

Finally, in section 7, in light of Theorem 3.3, we construct an algebraic Gysin sequence for Tor of $\mathbb{Z}[K]$ in analogy with the Gysin sequence of S^1 -fibration over a toric manifold.

2. Moment Angle Complexes and Toric Orbifolds

In this section, we review the basic facts about the moment angle complexes and various toric orbifolds to motivate our results.

2.1. Moment Angle Complexes. The *moment angle complex* \mathcal{Z}_K associated to a simplicial complex K was introduced by Buchstaber and Panov in [7] as a disc-circle decomposition of the Davis-Januszkiewicz universal space associated to a simplicial complex K [10] and it has been actively studied in *toric topology* and its connections to symplectic and algebraic geometry and combinatorics. For convenience, we use the following notation for the rest of the paper.

Notation 2.1. Let X, Y be the subsets of a set Z . For a subset $\sigma \subset [m]$, $X^\sigma \times Y^{[m] \setminus \sigma} \subset Z^m$ is the direct product of X and Y 's where i -th compont is X if $i \in \sigma$ and Y if $i \notin \sigma$.

For a simplicial complex K on vertices $[m] := \{1, \dots, m\}$, the moment angle complex $\mathcal{Z}_K \subset \mathbb{C}^m$ is defined as $\mathcal{Z}_K = \bigcup_{\sigma \in K} D^\sigma \times (\partial D)^{[m] \setminus \sigma}$ where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit disk and ∂D is its boundary circle. This space \mathcal{Z}_K carries a natural action of $T = U(1)^m$. It is originally proved in [10] that

$$H_T^*(\mathcal{Z}_K, \mathbb{Z}) \cong \mathbb{Z}[K] \quad \text{as graded rings over } \mathbb{Z}[T^*]. \quad (1)$$

Here $\mathbb{Z}[K]$ is the *Stanley-Reisner (face) ring* defined by $\mathbb{Z}[K] = \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle x_\sigma, \sigma \notin K \rangle}$ where $x_\sigma := \prod_{i \in \sigma} x_i$. With the identification $\mathbb{Z}[T^*] := H^*(BT, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$, the isomorphism is as graded algebras over the polynomial ring with $\deg x_i = 2$. For the details, we refer to Chapter 6 [6].

Baskakov-Buchstaber-Panov [3] also computed the ordinary cohomology of \mathcal{Z}_K :

$$H^*(\mathcal{Z}_K, \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[T^*]}^*(\mathbb{Z}[K], \mathbb{Z}) \quad \text{as graded rings.} \quad (2)$$

Here the grading on the right hand side is the total degree of bidegree coming from the (co)homological degree of Koszul complex and the degree of $\mathbb{Z}[K]$. More precisely

Definition 2.2. Let M be a graded $\mathbb{Z}[R^*] := \mathbb{Z}[u_1, \dots, u_n]$ -module. Let Λ be the exterior algebra generated by η_1, \dots, η_n with $\deg \eta_i = 1$. Let $\mathbb{Z}[R^*] \otimes^R \Lambda$ be the Koszul complex. Then $\text{Tor}_{\mathbb{Z}[R^*]}^{\mathbb{Z}[R^*]}(M, \mathbb{Z})$ is the homology of the complex $M \otimes_{\mathbb{Z}[R^*]} \mathbb{Z}[R^*] \otimes^R \Lambda$ where the degree is given by $\deg \eta_i = 1$. The complex $M \otimes_{\mathbb{Z}[R^*]} \mathbb{Z}[R^*] \otimes^R \Lambda$ is also a bigraded differential complex with bideg $(\alpha \otimes \xi_i) = [-1, 2] + [0, |\alpha|]$ where $\alpha \in M^{|\alpha|}$. The cohomological degree of Tor is defined to be the total degree of this bidegree and is denoted by the superscript as in $\text{Tor}_{\mathbb{Z}[R^*]}^*(M, \mathbb{Z})$.

Now it is natural to ask if $H_G^*(\mathcal{Z}_K, \mathbb{Z})$ can be computed by $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathcal{Z}_K, \mathbb{Z})$ where $G \subset T$ is a (possible disconnected) subgroup, $R := T/G$ and $\mathbb{Z}[R^*] := H^*(BR, \mathbb{Z}) \subset \mathbb{Z}[T^*]$. In Section 3, we show that

$$H_G^*(\mathcal{Z}_K, \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathcal{Z}_K, \mathbb{Z}) \quad \text{as graded } \mathbb{Z}[T^*]\text{-modules (Theorem 3.3).}$$

2.2. Partial Quotient of Moment Angle Complexes. When a subgroup $G \subset T$ acts on \mathcal{Z}_K locally freely, the *quotient stack* $[\mathcal{Z}_K/G]$ is a topological orbifold (as a stack), together with the residual action of $R := T/G$. Indeed, $[\mathcal{Z}_K/G]$ is topologically isomorphic to various toric “spaces”, including *quasi-toric orbifolds* defined by Poddar-Sarkar [27], symplectic compact toric orbifolds defined by Lerman-Tolman [23], and *algebraic toric orbifolds* defined by Borisov-Chen-Smith [4]. In the next section, we recall the construction of those spaces and see that the cohomology rings of all of these toric spaces in nice cases are computed as the quotient of the Stanley-Reisner ring $\mathbb{Z}[K]$.

In this section, we give a criteria for the local freeness of the action of a subgroup G of T on \mathcal{Z}_K and a remark about the cohomology of orbifolds as stacks.

Lemma 2.3. *Let n be the largest cardinality of a face in K . If a subgroup $G \subset T$ acts on \mathcal{Z}_K locally freely, then $\dim G \leq m - n$. Furthermore, if n is the cardinality of maximal faces in K (pure) and $\dim G = m - n$,*

then G acts on \mathcal{Z}_K locally freely if and only if $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$ surjects to $R := T/G$ for all maximal faces σ .

Proof. Let $\sigma \in K$ such that $|\sigma| = n$. Let $0_\sigma \in D^\sigma \times (\partial D)^{[m] \setminus \sigma}$ such that i -th component of 0_σ for $i \in \sigma$ is $0 \in D$. Then the stabilizer of 0_σ in T is $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$. Consider the map $\theta : G \times T_\sigma \rightarrow T, (g, t) \mapsto gt$. The kernel of this map θ is finite if and only if the stabilizer of 0_σ in G is finite. Therefore the local freeness of the G -action implies that the dimension of the image of θ is $\dim G + |\sigma| = \dim G + n$. Thus $\dim G \leq m - n$ since $\dim T = m$. To prove the latter claim, note that the local freeness of the G -action is equivalent to that the stabilizer of 0_σ in G is finite for each maximal face σ . Since $G = m - n$ and $\dim T_\sigma, G \cap T_\sigma$ is zero dimensional if and only if $T_\sigma \rightarrow R$ is surjective. \square

Remark 2.4. In [11], Edidin defined the integral cohomology of a stack and showed that if the stack is given as a global quotient stack $[M/G]$, then $H^*([M/G], \mathbb{Z})$ is canonically isomorphic to $H_G^*(M, \mathbb{Z})$. If G acts locally freely on M , the quotient stack $[M/G]$ is an orbifold. The cohomology of an orbifold $[M/G]$ as a stack is then $H_G(M, \mathbb{Z})$. On the other hand, the projection map $BG \times_G M \rightarrow M/G$ where M/G is the quotient topological space induces an isomorphism $H_G^*(M, \mathbb{Q}) \cong H^*(M/G, \mathbb{Q})$ since the fiber is “ \mathbb{Q} -acyclic”. If G acts freely on M , then $H^*([M/G], \mathbb{Z}) \cong H_G^*(M, \mathbb{Z}) \cong H^*(M/G, \mathbb{Z})$.

If L acts on M and G is a subgroup of L that acts on M locally freely, we have the action of $K := L/G$ on the orbifold $[M/G]$. In this case, there is an isomorphism of stacks $[[M/G]/K] \cong [M/L]$ and we can define $H_K^*([M/G], \mathbb{Z}) := H^*([M/L], \mathbb{Z}) = H_L^*(M, \mathbb{Z})$ (c.f. [28, 22]).

Remark 2.5. The following is a useful criteria for the connectedness of G . Let B be the integer matrix induced from the quotient map $T \rightarrow R$. Then G is connected if and only if $B : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ is surjective.

2.3. Quasi-toric Orbifolds. Quasi-toric manifolds are introduced and studied by Davis-Januszkiewicz [10] and its orbifold counterpart is studied by Poddar-Sarkar [27]. Let Δ be a simple polytope of dimensional n in \mathbb{R}^n . Let H_1, \dots, H_m be the facets of Δ and for a face $F_\sigma = \cup_{i \in \sigma} H_i$, let $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$. Define $\mathcal{Z}_\Delta := T \times \Delta / \sim$ where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ is contained in a relative interior of F_σ and $t_1 t_2^{-1} \in T_\sigma$ (c.f. Definition 6.1 [6]). It is known that \mathcal{Z}_Δ is a smooth manifold (c.f. Lemma 6.1 [6]). Let B be an integer $n \times m$ matrix such that for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding columns $\lambda_{i_1}, \dots, \lambda_{i_n}$ form a basis of \mathbb{Q}^n . By the assumption, B defines a surjective map $T \twoheadrightarrow R$ also denoted by B . Let G be the kernel of B . A *quasi-toric orbifold* for the pair (Δ, B) is defined as the quotient stack $[\mathcal{Z}_\Delta/G]$. Here note that the assumption on B is equivalent to the local freeness of the G -action on \mathcal{Z}_Δ (See Lemma 2.3). Since \mathcal{Z}_Δ is T -equivariantly homeomorphic to \mathcal{Z}_{K_Δ} where K_Δ is the simplicial complex associated to Δ (see Section 6.2. [6]), the quasi-toric orbifold $[\mathcal{Z}_\Delta/G]$ is topologically the quotient of the moment angle complex \mathcal{Z}_{K_Δ} by G . As a consequence and by Remark 2.4, the R -equivariant cohomology of the quasi-toric orbifold $[\mathcal{Z}_\Delta/G]$ is

$$H_R^*([\mathcal{Z}_\Delta/G], \mathbb{Z}) \cong H_T^*(\mathcal{Z}_\Delta, \mathbb{Z}) \cong H_T^*(\mathcal{Z}_{K_\Delta}, \mathbb{Z}) \cong \mathbb{Z}[K_\Delta].$$

The rational cohomology ring of the quasi-toric orbifold is computed by Poddar-Sarkar [27]:

Theorem 2.6 (Poddar-Sarkar). *If $[\mathcal{Z}_\Delta/G]$ is a quasi-toric orbifold, $H^*([\mathcal{Z}_\Delta/G], \mathbb{Q}) \cong \mathbb{Q}[K_\Delta]/\langle u_1, \dots, u_n \rangle$.*

Here $u_j = \sum_{i=1}^m B_{ji} x_i \in \mathbb{Z}[x_1, \dots, x_m]$ and we identify $H^*(BR; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$. A quasi-toric orbifold given by (Δ, B) is a *quasi-toric manifold* if and only if for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding columns $\lambda_{i_1}, \dots, \lambda_{i_n}$ of B form a \mathbb{Z} -basis. In this case, the isomorphism holds with \mathbb{Z} -coefficients:

Theorem 2.7 (Davis-Januszkiewicz [10]). *If $[\mathcal{Z}_\Delta/\mathbf{G}]$ is a quasi-toric manifold, $H^*([\mathcal{Z}_\Delta/\mathbf{G}], \mathbb{Z})$ is isomorphic to $\mathbb{Z}[K_\Delta]/\langle u_1, \dots, u_n \rangle$. Moreover $H^*([\mathcal{Z}_\Delta/\mathbf{G}], \mathbb{Z})$ has no \mathbb{Z} -torsion (which follows from [10, Theorem 3.1] and the fact that a quasi-toric manifold is closed and orientable).*

Since \mathcal{Z}_Δ is \mathbf{T} -equivariantly homeomorphic to \mathcal{Z}_{K_Δ} , we have

Corollary 2.8. *If K_Δ and B give a quasi-toric orbifold, then $H_{\mathbf{G}}^*(\mathcal{Z}_{K_\Delta}; \mathbb{Q}) \cong \mathbb{Q}[K_\Delta]/\langle u_1, \dots, u_n \rangle$. If K_Δ and B give a quasi-toric manifold, then $H_{\mathbf{G}}^*(\mathcal{Z}_{K_\Delta}; \mathbb{Z}) \cong \mathbb{Z}[K_\Delta]/\langle u_1, \dots, u_n \rangle$.*

2.4. Compact Symplectic Toric Orbifolds. Lerman-Tolman [23] classified compact symplectic toric (effective) orbifolds in terms of *labeled polytopes*. A labeled polytope (Δ, \mathbf{b}) is a rational simple polytope Δ in \mathbb{R}^n with each facet $H_i, i = 1, \dots, m$ is labeled by a positive integer \mathbf{b}_i . If β_i is the integral primitive inward normal vector to each facet H_i , then by assigning the integer matrix $B = [\mathbf{b}_1\beta_1, \dots, \mathbf{b}_m\beta_m]$, we obtain a quasi-toric orbifold given by (Δ, B) . Here the symplectic structure on $[\mathcal{Z}_\Delta/\mathbf{G}]$ comes from identifying \mathcal{Z}_Δ with the level set for the reduction of \mathbb{C}^m by the action of \mathbf{G} . A *compact symplectic toric manifold* is given by the labeled polytope such that $\mathbf{b}_i = 1, \forall i = 1, \dots, m$ and such that for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding primitive normal vectors $\beta_{i_1}, \dots, \beta_{i_n}$ of B form a \mathbb{Z} -basis. This is exactly the Delzant condition in the classification of compact symplectic manifolds.

2.5. Algebraic Toric (effective) orbifold (a.k.a. toric Deligne-Mumford stack). Let K be a pure simplicial complex on $[m]$. Define a fan Σ_K in \mathbb{R}^m where each cone is generated by the part of standard basis $\mathbf{e}_i, i \in \sigma$ for each $\sigma \in K$. The corresponding toric variety X_{Σ_K} is a smooth open subvariety of \mathbb{C}^m that is exactly the complement of subspace arrangements given by K (c.f. Chapter 8 [6]). There is a natural embedding of \mathcal{Z}_K into X_{Σ_K} and it is shown that

Proposition 2.9 ([6] Proposition 8.9). *There is a \mathbf{T} -equivariant deformation retract for $\mathcal{Z}_K \subset X_{\Sigma_K}$ and, in particular, $H_{\mathbf{G}}^*(\mathcal{Z}_K; \mathbb{Z}) \cong H_{\mathbf{G}_{\mathbb{C}}}^*(X_{\Sigma_K}; \mathbb{Z})$.*

The algebraic toric orbifolds studied by Borisov-Chen-Smith [4] are defined by the *stacky fan*. There they consider possibly noneffective orbifolds. In this paper, since we are interested in the effective case, we simplify the stacky fan and call it the *labeled fan*. A labeled fan (Σ, \mathbf{b}) is a simplicial fan in \mathbb{R}^n with each ray ρ_i is labeled by a positive integer \mathbf{b}_i where $i = 1, \dots, m$. Let K be the simplicial complex associated to Σ . Let β_i be the integral primitive generator of each ray ρ_i , define an integral $n \times m$ matrix $B := [\mathbf{b}_1\beta_1, \dots, \mathbf{b}_m\beta_m]$, and let \mathbf{G} be the kernel of the induced map of tori $B : \mathbf{T} \rightarrow \mathbf{R}$. The *toric Deligne-Mumford(DM) stack* associated to a labeled fan (Σ, \mathbf{b}) is defined as the quotient stack $\mathcal{X}_{\Sigma, \mathbf{b}} := [X_{\Sigma_K}/\mathbf{G}_{\mathbb{C}}]$ where $\mathbf{G}_{\mathbb{C}}$ is the complexification of \mathbf{G} .

A toric DM stack $\mathcal{X}_{\Sigma, \mathbf{b}}$ (or its labeled fan (Σ, \mathbf{b})) is *complete* if the fan is complete i.e. the union of cones is \mathbb{R}^n . A toric DM stack $\mathcal{X}_{\Sigma, \mathbf{b}}$ is a non-singular toric variety if and only if the labels $\mathbf{b}_i = 1$ and the fan Σ is non-singular, i.e. for each maximal cone, the primitive generators of the rays in the cone, $\beta_{i_1}, \dots, \beta_{i_n}$, form a \mathbb{Z} -basis. In this case, we call (Σ, \mathbf{b}) non-singular. For a non-singular and complete labeled fan, we have the following classical result:

Theorem 2.10 (Danilov[9], Jurkiewicz[21]). *If (Σ, \mathbf{b}) is non-singular and complete, then $H^*([\mathcal{X}_{\Sigma}/\mathbf{G}_{\mathbb{C}}], \mathbb{Z}) \cong \mathbb{Z}[K_\Sigma]/\langle u_1, \dots, u_n \rangle$. Furthermore, it has no \mathbb{Z} -torsion ([9] Theorem 10.8).*

In Proposition 3.7 [4], it is proved that the *coarse moduli space* (underlying algebraic variety) for $[X_{K_\Sigma}/\mathbf{G}_\mathbb{C}]$ is exactly the toric variety X_Σ (See also [8])¹. Thus for more general cases, one still has an isomorphism with \mathbb{Q} -coefficients:

Theorem 2.11 (Danilov[9]). *If (Σ, \mathbf{b}) is complete, then $H^*([X_{K_\Sigma}/\mathbf{G}_\mathbb{C}], \mathbb{Q}) \cong \mathbb{Q}[K]/\langle u_1, \dots, u_n \rangle$ where $u_j = \sum_{i=1}^m B_{ji}x_i$.*

Theorem 2.9 and Remark 2.4 therefore imply

Corollary 2.12. *If K and B are given by a complete labeled fan, then $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Q}) \cong \mathbb{Q}[K]/\langle u_1, \dots, u_n \rangle$. If K and B are given by a complete and non-singular labeled fan, then $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$.*

Note that the underlying combinatorial structures for quasi-toric orbifolds and toric DM stacks are both simplicial complexes. Any symplectic toric orbifold can be made into an algebraic one by taking the normal fans to the polytopes. However, not all quasi-toric orbifolds can be made algebraic. A toric DM stack associated to a *polytopal fan* can be made into a symplectic toric orbifold but there is a toric DM stack associated to a *non-polytopal fan*. Such a toric DM stack can not be realized even as a quasi-toric orbifold.

In the light of Theorem 3.3 and Corollary 2.8 and 2.12, it is natural to study the following question

Question 2.13. When is $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Z})$ a quotient of the Stanley-Reisner ring for a general subgroup \mathbf{G} ?

Our answer to this question is Theorem 5.1. Also we will see in Section 5 that when the \mathbf{G} -action is locally free and $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Q})$ is a quotient of Stanley-Reisner ring, then the dimension of \mathbf{G} must be maximal.

3. The Proof that $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[\mathbf{R}^*]}^*(\mathbb{Z}[K], \mathbb{Z})$

In this section, we prove that $H_\mathbf{G}^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to $\text{Tor}_{\mathbb{Z}[\mathbf{R}^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ as a graded module over $\mathbb{Z}[\mathbf{T}^*]$. The idea of the proof, especially to use the homological machinery developed in [13], was communicated to us by Franz. Throughout, we will use terminology found in [13].

We will use the following notation consistently throughout this paper unless otherwise specified:

Notation 3.1. Let K be a simplicial complex on $[m] := \{1, \dots, m\}$ (possibly with ghost vertices) and let \mathcal{Z}_K be the associated moment angle complex with the standard torus $\mathbf{T} := \text{U}(1)^m$ -action. Let $\mathfrak{t} := \text{Lie}(\mathbf{T}) = \mathbb{R}^m$ and let $\mathbf{N}_\mathbf{T} = \mathbb{Z}^m$ be the kernel of the exponential map $\mathfrak{t} \rightarrow \mathbf{T}$. Let $\mathbf{G} \subset \mathbf{T}$ be a (possibly disconnected) subgroup of dimension $m - n$ and let $\mathbf{R} := \mathbf{T}/\mathbf{G}$ be the quotient torus. We identify $\mathbf{R} \cong \text{U}(1)^n$ so that the quotient map $\mathbf{T} \rightarrow \mathbf{R}$ defines an integral $n \times m$ matrix B which is viewed as the surjective linear map $\mathfrak{t} \rightarrow \mathfrak{r} := \text{Lie } \mathbf{R}$.

Let $\mathbb{Z}[\mathbf{T}^*] := H^*(B\mathbf{T}; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$ where $\{x_j\}$ is the standard basis of $\mathbf{N}_\mathbf{T}^*$ and let $\mathbb{Z}[\mathbf{R}^*] := H^*(B\mathbf{R}; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$ where $\{u_i\}$ is the standard basis of $\mathbf{N}_\mathbf{R}^*$. We regard $\mathbb{Z}[\mathbf{R}^*]$ as a subring of $\mathbb{Z}[\mathbf{T}^*]$ so that $u_i := \sum_{j=1}^m B_{ij}x_j$. The Stanley-Reisner ring $\mathbb{Z}[K]$ is defined as the quotient of $\mathbb{Z}[\mathbf{T}^*]$ by the monomials corresponding to non-faces of K .

Definition 3.2. M is an $H^*(B\mathbf{R})$ -module up to homotopy if it is a module over the reduced cobar construction of $H^*(\mathbf{R})$ (Section 4 [14]).

¹Even if the toric orbifold is a non-trivial orbifold, its coarse moduli space could be a non-singular toric variety. The simplest example of such a case may be the weighted projective space $[\mathbb{C}P^1_2] = [\mathbb{C}^2 \setminus \{(0, 0)\}]/\mathbb{C}^\times$ where the action of \mathbb{C}^\times is weighted by $(1, 2)$. Its coarse moduli space is simply $\mathbb{C}P^1$.

Theorem 3.3. *Under Notation 3.1, there is an isomorphism of graded modules over $\mathbb{Z}[T^*]$.*

$$\Theta_R : H_G^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$$

where the cohomological grading on the right hand side is given in Definition 2.2.

Proof. We suppress the coefficient ring \mathbb{Z} . Given a map $p : Y \rightarrow BR$, the twisted tensor product $C^*(Y) \otimes^R H^*(R)$ is defined by (5.7) [13]. Proposition 5.2 in [13] states that there is a quasi-isomorphism of differential graded $H_*(R)$ -modules $\Phi_Y^* : C^*(Y) \otimes^R H^*(R) \rightarrow C^*(Y \times^{BR} ER)$ where $Y \times^{BR} ER$ is a pullback of $ER \rightarrow BR$ along $p : Y \rightarrow BR$.

Let $Y := ER \times_R (ET \times_G \mathcal{Z}_K)$ and $p : Y \rightarrow BR$ is defined as the composition of maps

$$ER \times_R (ET \times_G \mathcal{Z}_K) \xrightarrow{q_R} ET \times_T \mathcal{Z}_K \longrightarrow BT \rightarrow BR,$$

where q_R is the projection to the 2nd and 3rd components, the second map is the projection to the 1st component, and the last map is a classifying map for $B : T \rightarrow R$. We observe that p is obtained by taking the quotient of

$$ER \times (ET \times_G \mathcal{Z}_K) \rightarrow ET \times_G \mathcal{Z}_K \rightarrow ET/G \rightarrow ER$$

by the free actions of R on each space. This implies that $Y \times^{BR} ER = ER \times (ET \times_G \mathcal{Z}_K)$.

Now Proposition 5.2 [13] states that we have the quasi-isomorphism:

$$\Phi_Y^* : C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R) \rightarrow C^*(ER \times (ET \times_G \mathcal{Z}_K)).$$

The homology of the right hand side is $H_G^*(\mathcal{Z}_K)$. On the left hand side, since R acts on $ET \times_G \mathcal{Z}_K$ freely, the fibers of q_R are ER and therefore it induces a quasi-isomorphism of $H^*(BT)$ -modules up to homotopy

$$q_R^* : C^*(ET \times_T \mathcal{Z}_K) \rightarrow C^*(ER \times_R (ET \times_G \mathcal{Z}_K)),$$

i.e. it is a homomorphism of dg $C^*(BR)$ -modules such that after taking homology, it becomes an isomorphism of $H^*(BR)$ -modules. Theorem 1.1 [14] implies that $C^*(ET \times_T \mathcal{Z}_K)$ is formal as a $H^*(BT)$ -module up to homotopy, i.e there is a sequence of quasi-isomorphisms connecting $C^*(ET \times_T \mathcal{Z}_K)$ to $H^*(ET \times_T \mathcal{Z}_K)$ as dg modules over reduced cobar construction of $H^*(T)$, and therefore as dg modules over reduced cobar construction of $H^*(R)$. Since the operation to take the twisted tensor product $\otimes^R H^*(BR)$ and the homology of it preserves quasi-isomorphisms of $H^*(BR)$ -modules up to homotopy (c.f. Theorem 8.20, [26]), the map q_R^* induces a quasi-isomorphism

$$\widetilde{q_R^*} : C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R) \rightarrow C^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R).$$

and there is a sequence of quasi-isomorphisms connecting $C^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$ and $H^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$. Since the homology of the complex $H^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$ is $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$, we obtain the

isomorphism Θ_R . We summarize all in the following diagram:

$$\begin{array}{ccc}
 H_G^*(\mathcal{Z}_K) & \xleftarrow{\text{homology}} & C^*(ER \times (ET \times_G \mathcal{Z}_K)) \\
 \downarrow \Theta_R \cong & & \uparrow \Phi_Y^* \\
 & & C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R) \\
 & & \downarrow \bar{q}_R \\
 & & C^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R) \\
 & & \uparrow \text{seq of quasi-iso} \\
 \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}) & \xleftarrow{\text{homology}} & H^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R).
 \end{array}$$

The right vertical maps gives a sequence of quasi-isomorphisms of dg $H_*(R)$ -modules and at the both ends, we have the desired graded \mathbb{Z} -modules after taking homology.

To show that the map Θ_R is a homomorphism of modules over $\mathbb{Z}[T^*]$, it is sufficient to prove that the following diagram is commutative

$$\begin{array}{ccc}
 H_T^*(\mathcal{Z}_K, \mathbb{Z}) & \xrightarrow{\Theta} & \mathbb{Z}[K] \\
 \downarrow \iota_R^* & & \downarrow \phi_R \\
 H_G^*(\mathcal{Z}_K, \mathbb{Z}) & \xrightarrow{\Theta_R} & \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})
 \end{array}$$

where ϕ_R is the obvious map induced from the inclusion of Koszul complexes, ι_R^* is the pullback of the quotient map $\iota_R : ET \times_G \mathcal{Z}_K \rightarrow ET \times_T \mathcal{Z}_K$ and Θ is the isomorphism mentioned at (1) Section 2. Consider the map $\varphi : R \rightarrow \mathbf{1}$ where $\mathbf{1}$ is the trivial group. Since it satisfies the condition in Propostion 4.11 [13], the naturality stated in Proposition 5.2 [13] implies that the following diagram commutes

$$\begin{array}{ccc}
 C^*(ET \times_T \mathcal{Z}_K) & \xleftarrow{\quad} & C^*(ET \times_T \mathcal{Z}_K) \otimes^1 H^*(\mathbf{1}) \\
 \downarrow \bar{q}_R^* & & \downarrow \bar{\phi}_R \\
 C^*(ER \times (ET \times_G \mathcal{Z}_K)) & \xleftarrow{\quad} & C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R)
 \end{array}$$

where \bar{q}_R^* is the pullback of the projection $\bar{q}_R : ER \times (ET \times_G \mathcal{Z}_K) \rightarrow ET \times_T \mathcal{Z}_K$ and $\bar{\phi}_R$ is the map induced by q_R and $\varphi : R \rightarrow \mathbf{1}$. After taking the homology, \bar{q}_R^* and $\bar{\phi}_R$ naturally coincide with ι_R^* and ϕ_R respectively. \square

Remark 3.4. Let X be any topological space with the T -action such that $C^*(ET \times_T X; \mathbb{Z})$ is formal as $H^*(BT; \mathbb{Z})$ -modules up to homotopy. Then Theorem 3.3 also holds for X . Namely, the above proof can be identically applied to this case and gives the isomorphism $\Theta_R : H_G^*(X; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(H_T^*(X; \mathbb{Z}), \mathbb{Z})$.

In terms of toric orbifolds discussed in Section 2, we have

Corollary 3.5. *Let X is a quasi-toric orbifold or effective toric Deligne-Mumford stack with associated K, R and T in the sense of Section 2. There is an isomorphism of graded modules over $\mathbb{Z}[T^*]$*

$$H^*(X; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}).$$

Remark 3.6. Theorem 3.3 (or Corollary 3.5) generalizes several results that have been proved. For example:

- (1) In case $G = T$, Theorem 3.3 states that the T -equivariant cohomology of \mathcal{Z}_K is the Stanley-Reisner ring of K , which is well-known (c.f.[6]). In case $G = 1$, then one recovers that the ordinary cohomology of \mathcal{Z}_K is the Tor-algebra of $\mathbb{Z}[K]$ over $\mathbb{Z}[T^*]$ (Theorem 7.6 [6]). One may therefore view this result as interpolating between these extreme cases.
- (2) If \mathcal{Z}_K/G is a quasi-toric manifold, then one recovers Theorem 7.37 [6].
- (3) When $X_\Sigma := X_{\Sigma_K}/G_{\mathbb{C}}$ is the coarse moduli for a toric orbifold, one recovers Theorem 1.2 [14]:

$$H^*(X_\Sigma, \mathbb{Q}) \cong H_{G_{\mathbb{C}}}^*(X_{\Sigma_K}, \mathbb{Q}) = H_G^*(\mathcal{Z}_K, \mathbb{Q}) \cong \mathrm{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}) \otimes \mathbb{Q}.$$

4. Basics from commutative algebra

In this section, we collect some definitions and basic properties of graded modules over a polynomial ring and discuss the relations among them. Throughout this section, R will be a polynomial ring in variables u_1, \dots, u_n (generated in degree 2) over $k = \mathbb{Z}$ or \mathbb{Q} , and M will be a graded R -module (though not necessarily finitely generated). We will denote the ideal of R generated by polynomials of positive degree by R_+ .

Below we give brief names to several properties of R -modules so that we can refer to them later.

Definition 4.1. For M and R as above, one says that:

- (k1) M is *free over R* if $M \cong \bigoplus_{e \in E} R \cdot e$ and $R \cdot e \cong R$ as graded R -modules where E is a subset of M ;
- (k2) M is *flat over R* if $\mathrm{Tor}_{>0}^R(M, N) = 0$ for any (f.g.) module N ;
- (k3) M is *torsion-free over R* if there is no torsion over R ($x \in M$ is a torsion element over R if $\cdot x : R \rightarrow R$ has non-trivial kernel);
- (k4) M is a *big Cohen-Macaulay R -module* if $\mathrm{Tor}_1^R(M, k) = 0$.

In general, one has the following implications.

$$(k1) \Rightarrow (k2) \Rightarrow (k3), (k4)$$

We will see that in the case of interest to us, (k4) has the usual meaning in terms of regular sequences; see Proposition 4.6.

Definition 4.2. A non-zero element $r \in R$ is *M -regular* if $0 \rightarrow M \xrightarrow{r} M$ is exact. A sequence of elements $f_1, \dots, f_c \in R$ is an *M -regular sequence* if, for each $i \leq c$, f_i is $(M/(f_1, \dots, f_{i-1})M)$ -regular.

Remark 4.3. We call the condition (k4) in Definition 4.1 *big Cohen-Macaulay* because $\mathrm{Tor}_1^R(M, k) = 0$ is the same as saying that there exists a system of parameters of R_+ (namely, the variables of R ; see Corollary 4.12) that is an M -regular sequence. The ‘big’ terminology is a reference to the fact that M need not be finitely generated; see [5, Chapter 8] ².

Definition 4.4. Let R_+ be the ideal generated by the positive degree elements of R , and suppose that $M \neq IM$. One defines the *depth* (as well as *grade*) of M over R by

$$\mathrm{depth}_R(M) := \mathrm{grade}(I, M) := \min\{i \mid \mathrm{Ext}_R^i(R/I, M) \neq 0\}. \quad (3)$$

If $M = IM$, one sets $\mathrm{depth}_R(M) = \infty$. When M is finitely generated over R , this definition is the usual definition of the depth of M and it is the length of maximal M -regular sequence in R_+ [5].

²In the reference [5], it is mentioned that the existence of a big Cohen-Macaulay module over a local ring R is an open problem. This question is not interesting for R , since it is a (non-local) Cohen-Macaulay ring.

Our goal for this section is to show that $\text{depth}_R(M)$ is the length of the longest M -regular sequence in R_+ when M is only assumed to be finitely generated over some homomorphic image of R , and the R action on M factors through this homomorphic image. This is recorded in the following proposition.

Proposition 4.5. *Let $S = k[x_1, \dots, x_m]$, M be a finitely generated graded S -module, and $\varphi : R \rightarrow S$ be a graded ring homomorphism (so that M is hence a graded R -module via φ). Then all maximal M -regular sequences in R_+ have the same length $\text{depth}_R(M)$.*

This proposition is a special case of the following proposition, whose proof will come after some lemmas.

Proposition 4.6. *Let $\varphi : R \rightarrow S$ be a homomorphism of Noetherian rings, M a finitely generated S -module and I an ideal of R with $IM \neq M$. Then*

$$\text{grade}(I, M) := \min\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}$$

agrees with the length of all maximal M -regular sequences in I .

The fact that allows one to extend the usual finitely generated setup to the generality above is the following lemma.

Lemma 4.7. *Let $\varphi : R \rightarrow S$ be a homomorphism of graded Noetherian rings, M a finitely generated S -module, and N a finitely generated R -module. Then $\text{Hom}_R(N, M) = 0$ if and only if $\text{Ann}_R(N)$ contains a non-zero divisor on M .*

Remark 4.8. Before starting on the proof, let us remark that in this case, the set of associated primes of M over R is finite, even though M may not be finitely generated; see [25, Exercise 6.9]. Indeed, one sees this by taking a primary decomposition of M over S , which is also a primary decomposition over R , since the R -annihilator of an S -module is the preimage of the annihilator in S (and the inverse image of a primary ideal is primary).

Proof of Lemma 4.7. Suppose that $x \in \text{Ann}_R(N)$ is a non-zero divisor. Then for any $\psi \in \text{Hom}_R(N, M)$, we have

$$x\psi(n) = \psi(xn) = \psi(0) = 0, \forall n \in N.$$

Since x is a non-zero divisor on M , we have $\psi(n) = 0$.

Now assume that $\text{Ann}_R(N)$ consists of zero-divisors on M . As mentioned in Remark 4.8, the set of associated primes of M over R is finite. Since $\text{Ann}_R(N)$ consists of zerodivisors, it is contained in the (finite) union of all associated primes of M . Therefore, we can apply the Prime Avoidance Lemma to get $\text{Ann}_R(N) \subset \mathfrak{p}$ for some associated prime \mathfrak{p} of M over R . We then have the following non-trivial map:

$$N_{\mathfrak{p}} \twoheadrightarrow N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}} \twoheadrightarrow k(\mathfrak{p}) \hookrightarrow M_{\mathfrak{p}},$$

where $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Thus, since N is finitely generated over R , $\text{Hom}_R(N, M)_{\mathfrak{p}} = \text{Hom}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$, which gives $\text{Hom}_R(N, M) \neq 0$. \square

We also record without proof a basic fact from the homological algebra of commutative rings.

Lemma 4.9. *Let $\varphi : R \rightarrow S$ be a homomorphism of Noetherian rings, M a finitely generated S -module, and N a finitely generated R -module. If (x_1, \dots, x_r) is a regular sequence in $\text{Ann}_R(N)$ for M , then*

$$\text{Hom}_R(N, M/(x_1, \dots, x_r)M) = \text{Ext}_R^r(N, M).$$

\square

Proof of Proposition 4.6. Let (x_1, \dots, x_r) be a maximal M -regular sequence in I . By Lemma 4.9,

$$\mathrm{Ext}_R^i(R/I, M) \cong \mathrm{Hom}_R(R/I, M/(x_1, \dots, x_i)M).$$

If $i < r$, then x_{i+1} is a non-zero-divisor in $M/(x_1, \dots, x_i)M$, therefore by Lemma 4.7, $\mathrm{Ext}_R^i(R/I, M) = 0$. Since (x_1, \dots, x_r) is maximal, I doesn't contain any non-zero-divisor in $M/(x_1, \dots, x_r)M$. Thus $\mathrm{Ext}_R^r(R/I, M) \neq 0$. This proves the claim. \square

One has the following well known characterization of (k4) in terms of M -regular sequences due to Serre [29, Chapter IV.A].

Proposition 4.10. *Let u_1, \dots, u_n be a homogeneous minimal generating set of R_+ . Suppose that R and M satisfy the hypotheses of Proposition 4.5. Then the following properties are equivalent:*

- a) $H_p(\mathbf{u}, M) = 0$ for $p \geq 1$.
- b) $H_1(\mathbf{u}, M) = 0$.
- c) *The sequence u_1, \dots, u_n is M -regular.*

Here, $H_p(\mathbf{u}, M)$ denotes the p^{th} Koszul homology of the sequence u_1, \dots, u_n with coefficients in M .

Proof. The proof that appears in *op. cit.* uses standard techniques of the Koszul complex which hold even for modules which are not finitely generated over R , together with Nakayama's lemma for finitely generated modules over a Noetherian local ring. Since a version of Nakayama's lemma holds for graded modules that satisfy our hypothesis, Serre's original argument remains valid. \square

Corollary 4.11. *Let u_1, \dots, u_n be a homogeneous minimal generating set of R_+ , and suppose that R and M satisfy the hypothesis of Proposition 4.6. Then $\mathrm{Tor}_1^R(M, k) = 0$ if and only if (u_1, \dots, u_n) is a regular sequence for M .*

Proof. The Koszul complex on u_1, \dots, u_n resolves k over R , and hence one can use its homology to compute the Tor modules. Now appeal to the previous proposition. \square

Propositions 4.11 and 4.5 show that in the setup of 4.5, big Cohen-Macaulayness of M has the usual meaning in terms of the maximal length of an M -regular sequence.

Corollary 4.12. *In the setup of Proposition 4.5, one has that $\mathrm{Tor}_1^R(M, k) = 0$ if and only if $\mathrm{depth}_R(M) = n$.*

5. Properties of $\mathbb{Z}[K]$ as an algebra over $\mathbb{Z}[\mathbb{R}^*]$

In this section, we start with the characterization of the big Cohen-Macaulayness (k4), and then discuss the freeness (k1) and the torsion-freeness (k3), for $\mathbb{Z}[K]$ as a ring over $\mathbb{Z}[\mathbb{R}^*]$. In the rest of the paper, we use Notation 3.1 unless otherwise specified.

5.1. Big Cohen-Macaulayness. The following theorem is a variant of Theorem 1.1 [16] and Lemma 5.1 in [15]. The differences from them are that the T-CW complex $EG \times_G \mathcal{Z}_K$ is not finite and that we consider the cohomology of the quotient stack $[\mathcal{Z}_K/\mathbb{G}]$ instead of the one of the underlying topological space \mathcal{Z}_K/\mathbb{G} .

Theorem 5.1. *Let $\iota_R : ET \times_G \mathcal{Z}_K \rightarrow ET \times_T \mathcal{Z}_K$ be the quotient map by the \mathbb{R} -action. The following are equivalent:*

- (1) $\mathbb{Z}[K]$ is big Cohen-Macaulay over $\mathbb{Z}[\mathbb{R}^*]$, i.e. $\mathrm{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$.
- (2) $\iota_R^* : H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is surjective.

$$(3) H_G^{odd}(\mathcal{Z}_K; \mathbb{Z}) = 0.$$

Proof. Since $\text{Tor}_0^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$, Theorem 3.3 implies that (2) is equivalent to the vanishing of $\text{Tor}_{i>0}^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z})$, which is actually equivalent to (1) by Proposition 2.3 [16]. (2) implies (3) because $H_T^*(\mathcal{Z}_K; \mathbb{Z})$ has only even degree classes. Now (3) implies that the Serre spectral sequence for the Borel construction for the residual \mathbb{R} -action for $ET \times_G \mathcal{Z}_K$ degenerates at E_2 level and hence the pullback of the fiber inclusion $ET \times_G \mathcal{Z}_K \hookrightarrow ER \times_R (ET \times_G \mathcal{Z}_K)$ is surjective. This pullback can be identified as ι^* and thus (3) implies (2) (See also Lemma 5.1 [15] and its proof). \square

Again it is worth noting that Theorem 5.1 for any \mathbb{T} -space X that satisfies the formality (see Remark 3.4) and such that $H_T^{odd}(X; \mathbb{Z})$.

Remark 5.2. By Theorem 3.3, ι_R^* is surjective if and only if $\Theta : H_G^*(\mathcal{Z}_K, \mathbb{Z}) \cong \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$.

Remark 5.3. Theorem 5.1 holds after replacing \mathbb{Z} by \mathbb{Q} .

Corollary 5.4. *If $X = [\mathcal{Z}_K/\mathbb{G}]$ is a quasi-toric orbifold or effective toric Deligne-Mumford stack, then the following are equivalent*

- (1) $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$.
- (2) $H^*(X; \mathbb{Z})$ is the quotient of $\mathbb{Z}[K]$ by u_1, \dots, u_n .
- (3) $H^{odd}(X; \mathbb{Z}) = 0$.

5.2. Freeness. The following theorem is analogous to Lemma 6.1 [15].

Proposition 5.5. $\mathbb{Z}[K]$ is free over $\mathbb{Z}[\mathbb{R}^*]$ if and only if $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ has no \mathbb{Z} -torsion and has only even degree.

Proof. If $H_G(\mathcal{Z}_K, \mathbb{Z})$ has only even degree, then $\iota_R^* : H_T^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K, \mathbb{Z})$ is surjective by Theorem 5.1. The surjectivity implies that $H_G^r(\mathcal{Z}_K, \mathbb{Z})$ has finite rank for each r and is actually a finitely generated free \mathbb{Z} -module if it has no \mathbb{Z} -torsion. The Leray-Hirsch Theorem (c.f. Theorem 4D.1 [18]) can be applied to the fiber bundle $ER \times_R (ET \times_G \mathcal{Z}_K) \rightarrow BR$ where the pullback along the fiber $ET \times_G \mathcal{Z}_K$ can be identified with ι^* and therefore we have the isomorphism $\mathbb{Z}[\mathbb{R}^*] \otimes_{\mathbb{Z}} H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]$. Since $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is a free \mathbb{Z} -module, $\mathbb{Z}[K]$ is a free $\mathbb{Z}[\mathbb{R}^*]$ -module.

On the other hand, the freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[\mathbb{R}^*]$ implies $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ and so ι_R^* is surjective by Theorem 5.1. Thus $H_G(\mathcal{Z}_K, \mathbb{Z}) \cong H_T(\mathcal{Z}_K, \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{R}^*]} \mathbb{Z}$. By freeness, we can write $H_T(\mathcal{Z}_K; \mathbb{Z}) \cong \bigoplus_{e \in E} \mathbb{Z}[\mathbb{R}^*]e$ as a free $\mathbb{Z}[\mathbb{R}^*]$ -module, where e 's are even degree classes. Then $H_T(\mathcal{Z}_K; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{R}^*]} \mathbb{Z} \cong \bigoplus_{e \in E} \mathbb{Z}e$. Thus there are no \mathbb{Z} -torsions and no odd degree classes. \square

The same proof as above proves the following (see also Remark 5.3):

Proposition 5.6. $\mathbb{Q}[K]$ is free over $\mathbb{Q}[\mathbb{R}^*]$ if and only if $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree classes.

With this proposition, together with the local freeness of the \mathbb{G} -action, we can also prove the following lemma.

Lemma 5.7. *Suppose that the \mathbb{G} -action on \mathcal{Z}_K is locally free. If $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree, then $\mathbb{Q}[K]$ is finitely generated over $\mathbb{Q}[\mathbb{R}^*]$.*

Proof. Since the \mathbb{G} -action on the smooth variety X_{Σ_K} defined in Section 2.5 is locally free, we have the differentiable orbifold $[X_{\Sigma_K}/\mathbb{G}]$. By the construction of de Rham cohomology for differentiable orbifolds, c.f.

§ 2.1 [1], $H^*([X_{\Sigma_K}/G]; \mathbb{R})$ is finitely dimensional. Since $\mathcal{Z}_K \hookrightarrow X_{\Sigma_K}$ is a T -equivariant deformation retract, $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is also finite dimensional. On the other hand, by Proposition 5.6, if $H_G^*(\mathcal{Z}_K, \mathbb{Q})$ has no odd degree, then $\mathbb{Q}[K]$ is free over $\mathbb{Q}[R^*]$. Since $H_G^*(\mathcal{Z}_K; \mathbb{Q}) \cong H_T^*(\mathcal{Z}_K; \mathbb{Q}) \otimes_{\mathbb{Q}[R^*]} \mathbb{Q}$, the finiteness of $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ implies that $H_T^*(\mathcal{Z}_K; \mathbb{Q})$ is finitely generated over $\mathbb{Q}[R^*]$. \square

5.3. Torsion-freeness. First we observe the following equivalence.

Lemma 5.8. *Then $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[R^*]$ if and only if $\mathbb{Z}[K] \otimes \mathbb{Q}$ is torsion-free over $\mathbb{Q}[R^*]$.*

Proof. Suppose that $f \neq 0$ is a torsion element in $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$, i.e. there is $g \in \mathbb{Z}[R^*]$ such that $fg = 0$ in $\mathbb{Z}[K]$. Since $\mathbb{Z}[K]$ is free over \mathbb{Z} , $f \neq 0$ in $\mathbb{Q}[K]$. Therefore f is also a torsion in $\mathbb{Q}[K]$ over $\mathbb{Q}[R^*]$. On the other hand, suppose that $f \neq 0$ is a torsion element of $\mathbb{Q}[K]$ over $\mathbb{Q}[R^*]$. Let $g \in \mathbb{Q}[R^*]$ such that $fg = 0$ in $\mathbb{Q}[K]$. Let a be the product of denominators of the coefficients of f and b be the product of denominator of coefficients of g . Then the pair of $af \in \mathbb{Z}[K]$ and $bg \in \mathbb{Z}[R^*]$ gives a torsion of $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$. \square

Theorem 5.9. *If $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree classes, then $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[R^*]$.*

Proof. By Proposition 5.6, $\mathbb{Q}[K]$ is free over $\mathbb{Q}[R^*]$, therefore it is torsion-free over $\mathbb{Q}[R^*]$. We conclude that $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[R^*]$ by Lemma 5.8. \square

If $[\mathcal{Z}_K/G]$ is a quasi-toric orbifold or a complete toric DM stack (Section 2), $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree classes by Theorem 2.6 and 2.11. Thus we have

Corollary 5.10. *If $[\mathcal{Z}_K/G]$ is a quasi-toric orbifold or a complete toric DM stack, then $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[R^*]$.*

Remark 5.11. The converse of Theorem 5.9 is not true. The direct product of weighted projective spaces is a complete toric DM stack and its cohomology has odd degree classes. See Example 6.2.

The following proposition shows that the vanishing of odd classes in $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ implies that the size of G is maximal and it is analogous to Proposition 5.2 [15].

Proposition 5.12. *Let n' be the largest cardinality of a face of K . Suppose that G acts on \mathcal{Z}_K locally freely. If $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree, then $\dim R = n'$. Furthermore for any subgroup $U \subset G$ such that $\dim U < \dim G$, $\mathbb{Q}[K]$ has a torsion over $\mathbb{Q}[\tilde{R}^*]$ where $\tilde{R} := T/U$.*

Proof. By Proposition 5.7, $\mathbb{Q}[K]$ is finitely generated over $\mathbb{Q}[R^*]$ and hence over $\mathbb{Q}[\tilde{R}^*]$. Moreover $\mathbb{Q}[K]$ is free over $\mathbb{Q}[R^*]$ by Proposition 5.6 and so $\text{Ann}_{\mathbb{Q}[R^*]} \mathbb{Q}[K] = 0$. Thus we have

$$n' = \dim \mathbb{Q}[K] = \dim \mathbb{Q}[R^*] = \dim \frac{\mathbb{Q}[\tilde{R}^*]}{\text{Ann}_{\mathbb{Q}[\tilde{R}^*]} \mathbb{Q}[K]}.$$

Thus $\dim R = \dim \mathbb{Q}[R^*] = n'$ and $\dim \mathbb{Q}[\tilde{R}^*] = \dim \tilde{R} > \dim R$ implies $\text{Ann}_{\mathbb{Q}[\tilde{R}^*]} \mathbb{Q}[K] \neq 0$. \square

The above proposition doesn't allow us to construct a torsion element explicitly. Below we show a way to find one for the case of toric manifolds .

5.3.1. How to find a torsion element. We will use the GKM description of the equivariant cohomology of a toric manifold. Let Δ be an n -dimensional Delzant polytope, i.e. a labeled polytope associated to a toric manifold. Let H_1, \dots, H_m be the facets of Δ . As in Section 2.3 and 2.4, $T = U(1)^m$, $R = U(1)^n$ and $B : T \rightarrow R$ is given by the $n \times m$ integral matrix $B := [\beta_1, \dots, \beta_m]$ where β_i is the inward primitive normal vector for H_i . We adopt Notation 3.1. Let v_1, \dots, v_d be the vertices of Δ and let e_{ij} be the edge connecting v_i and v_j .

The R -equivariant cohomology of X_Δ coincides with the T -equivariant cohomology of \mathcal{Z}_Δ and it is the Stanley-Reisner ring $\mathbb{Z}[K_\Delta]$ for the associated simplicial complex K_Δ . Since Δ is Delzant, $\mathbb{Z}[K_\Delta]$ is free over $\mathbb{Z}[R^*]$ (c.f. Theorem 2.7, Proposition 5.5).

The injectivity theorem of the Hamiltonian R -action on X_Δ states that pulling back to fixed points gives the injective map of $\mathbb{Q}[R^*]$ -algebras:

$$\mathbb{Q}[K_\Delta] \xrightarrow{\Phi} \bigoplus_{i=1}^d \mathbb{Q}[u_1, \dots, u_n]$$

and the GKM theorem states that the image of Φ is given by

$$\text{GKM}(\Delta) := \left\{ f := (f_1, \dots, f_d) \in \bigoplus_{i=1}^d \mathbb{Q}[u_1, \dots, u_n] \mid \alpha_{ij}(f_i - f_j) \text{ for each edge } e_{ij} \right\}$$

where α_{ij} is a linear polynomial in u_1, \dots, u_n associated to each edge e_{ij} in the GKM theorem. The following proposition describes the image of $x_i \in \mathbb{Q}[K_\Delta]$ explicitly in terms of the matrix B .

Proposition 5.13. *Let $v := H_{i_1} \cap \dots \cap H_{i_n}$ be a vertex of Δ and $\sigma_v = \{i_1 < \dots < i_n\}$. Let $B_v := [\beta_{i_1}, \dots, \beta_{i_n}]$. Let α_r^v be the r -th row of B_v^{-1} . Then*

$$\Phi(x_i)_v = \begin{cases} \alpha_r^v \cdot (u_1, \dots, u_n)^T & \text{if } i = i_r \text{ for some } r = 1, \dots, n \\ 0 & \text{if otherwise.} \end{cases}$$

Proof. Recall that x_i is the first Chern class of the pullback of the circle bundle $L_i := ET \times_T T_i \rightarrow BT$ along $ET \times_T \mathcal{Z}_\Delta \rightarrow BT$ where $T_i = U(1)^{\{i\}} \times \{1\}^{[m] \setminus \{i\}} \subset T$ (see Notation 2.1). The Borel space of the fixed point is $ET \times_T ((T \times v)/T_{\sigma_v})$ as a subspace of $ET \times_T \mathcal{Z}_\Delta$ (See the definition of \mathcal{Z}_Δ in Section 2.3). Thus $\Phi(x_i)_v$ is the first Chern class of the pullback of the line bundle L_i along the projection $\pi : ET \times_T ((T \times v)/T_{\sigma_v}) \rightarrow BT$. Since $L_i \cong BT_{[m] \setminus \{i\}} \times ET_i$ and $ET \times_T ((T \times v)/T_{\sigma_v}) \cong BT_{\sigma_v} \times ET_{[m] \setminus \sigma_v}$, it is clear that, if $i \notin \sigma_v$, then $\pi^* L_i$ is trivial.

Now suppose that $i = i_r$ for some $r = 1, \dots, n$. Recall from Notation 3.1 that $u_k = \sum_{j=1}^m B_{kj} x_j$ where $B = (B_{kj})$. We observe that, for each $k = 1, \dots, n$, the i_k -th column of $B_v^{-1} \cdot B$ is $(0, \dots, 0, 1, 0, \dots, 0)^T$ where 1 is located at k -th entry. Therefore the r -th row of $B_v^{-1} \cdot B \cdot (x_1, \dots, x_m)^T$ is $x_{i_r} + \sum_{j \in [m] \setminus \sigma_v} c_j x_j$ for some integers c_j . Thus

$$\alpha_r^v \cdot (u_1, \dots, u_n)^T = \alpha_r^v \cdot B \cdot (x_1, \dots, x_m)^T = x_{i_r} + \sum_{j \in [m] \setminus \sigma_v} c_j x_j.$$

From the previous argument, we have $\Phi(x_i)_v = \Phi(x_i + \sum_{j \in [m] \setminus \sigma_v} a_j x_j)_v$ for any $a_j \in \mathbb{Q}$. Also since Φ is a $\mathbb{Q}[R^*]$ -algebra homomorphism, $\Phi(u) = (u, \dots, u)$ for any linear combination u of u_1, \dots, u_n . Thus

$$\Phi(x_i)_v = \Phi \left(x_i + \sum_{j \in [m] \setminus \sigma_v} c_j x_j \right) \Big|_v = \Phi \left(\alpha_r^v \cdot (u_1, \dots, u_n)^T \right) \Big|_v = \alpha_r^v \cdot (u_1, \dots, u_n)^T.$$

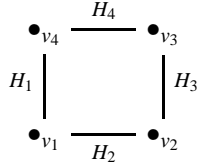
□

Remark 5.14. The row vector α_r^v is perpendicular to β_{i_k} , $k \neq r$ and is parallel to the edge $\cap_{k \neq r} H_k$. Furthermore since the inner product of α_r^v and β_{i_r} is 1, we can conclude that it is the primitive vector parallel to the edge and pointing out from v along the edge.

Corollary 5.15. *Using the notation above, let $U \subset G$ be a subtorus of dimension $\dim G - 1$. Let $\tilde{R} := T/U$ and $\mathbb{Z}[\tilde{R}^*] = \mathbb{Z}[u_1, \dots, u_n, u_{n+1}]$. Let $v = H_{i_1} \cap \dots \cap H_{i_n}$ be a vertex of Δ . Then $x_{i_1} \cdots x_{i_n}$ is a torsion element in $\mathbb{Q}[K_\Delta]$ over $\mathbb{Q}[\tilde{R}^*]$.*

Proof. First note that $x_{i_1} \cdots x_{i_n}$ is a none zero element in $\mathbb{Z}[K]$ and, by the above proposition, $\Phi(x_{i_1} \cdots x_{i_n})|_w = 0$ for all $w \neq v$. Thus we have $(u_{n+1} - \sum_{i=1}^n a_i u_i) \cdot \Phi(x_{i_1} \cdots x_{i_n}) = 0$ if we let $\Phi(u_{n+1})|_v = \sum_{i=1}^n a_i u_i$. By the injectivity theorem, $(u_{n+1} - \sum_{i=1}^n a_i u_i) \cdot (x_{i_1} \cdots x_{i_n}) = 0$. Since $u_{n+1} - \sum_{i=1}^n a_i u_i$ is none-zero by definition, we conclude that $x_{i_1} \cdots x_{i_n}$ is a torsion element over $\mathbb{Z}[\tilde{R}^*]$. \square

Example 5.16. Let Δ be the Delzant polytope that is the unit square where H_i 's are facets and v_j 's are vertices:



Let $T = U(1)^4$, $R = U(1)^2$ and the map $B : T \rightarrow R$ given by the matrix $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$. The kernel of B is $G = \{(t, s, t^{-1}, s^{-1})\}$. The corresponding toric manifold is $\mathbb{CP}^1 \times \mathbb{CP}^1$. By Proposition 5.13, the injectivity map $\Phi : \mathbb{Q}[K_\Delta] \rightarrow \bigoplus_{i=1}^4 \mathbb{Q}[u_1, u_2]$ is given by

$$\Phi(x_1) = (u_1, 0, 0, u_1), \quad \Phi(x_2) = (u_2, u_2, 0, 0), \quad \Phi(x_3) = (0, -u_1, -u_1, 0), \quad \Phi(x_4) = (0, 0, -u_2, -u_2).$$

where $u_1 = x_1 - x_3$, $u_2 = x_2 - x_4$. The GKM theorem states that the image can be described by GKM condition:

$$\text{Im}(\Phi) = \left\{ (f_1, f_2, f_3, f_4) \in \bigoplus_{i=1}^4 \mathbb{Q}[u_1, u_2] \mid u_1 | f_1 - f_2, \quad u_2 | f_2 - f_3, \quad u_1 | f_3 - f_4, \quad u_2 | f_4 - f_1 \right\}.$$

Let $U := \{(1, s, 1, s^{-1})\} \subset G$ be the subtorus and $\tilde{R} = T/U$. By Proposition 5.12, $\mathbb{Q}[K_\Delta]$ has torsion elements as a module over $\mathbb{Q}[\tilde{R}^*] = \mathbb{Q}[u_1, u_2, u_3]$, where we can take $u_3 = x_2 + x_3 - x_4$ and so $\Phi(u_3) = (u_2, -u_1 + u_2, -u_1 + u_2, u_2)$.

By Corollary 5.15, $x_1 x_2$ is a torsion element. Indeed, $\Phi(x_1 x_2) = (u_1 u_2, 0, 0, 0)$. Consider $u_3 - u_2 \in \mathbb{Q}[\tilde{R}^*]$. We have $\Phi(u_3 - u_2) = (0, -u_1, -u_1, 0)$. Since $(0, -u_1, -u_1, 0) \cdot (u_1 u_2, 0, 0, 0) = (0, 0, 0, 0)$ and by the injectivity, $(u_3 - u_2) \cdot x_1 x_2 = 0$ in $\mathbb{Q}[K_\Delta]$. Thus $x_1 x_2 \in \mathbb{Q}[K_\Delta]$ is a torsion element over $\mathbb{Q}[\tilde{R}^*]$. It is not hard to see what are doing here works equally well for general Delzant polytope Δ in any dimension.

5.4. An injectivity theorem and freeness. Suppose that a connected subtorus G of T acts on \mathcal{Z}_K locally freely and consider the a torus W such that $G \subsetneq W \subset T$ and $m' := \dim W$. Let $F \subset \mathcal{Z}_K$ be the set of $(m - n)$ -dimensional W -orbits. In this section, we discuss the injectivity of

$$H_W(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_W(F; \mathbb{Z}).$$

We have the following injectivity result for a W -action on \mathcal{Z}_K when $[\mathcal{Z}_K/G]$ is a symplectic compact toric orbifold.

Theorem 5.17. *Suppose that $[\mathcal{Z}_K/\mathbb{G}]$ is a symplectic compact toric orbifold corresponding to a labeled polytope (Δ, \mathbf{b}) . Suppose that the stabilizer of any point $x \in \mathcal{Z}_K$ in W is connected. Then $H_{\mathbb{T}}(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_W(F; \mathbb{Z})$ is injective. In particular, $H_{\mathbb{T}}(\mathcal{Z}_K; \mathbb{Z})$ is free over $\mathbb{Z}[(\mathbb{T}/W)^*]$.*

Proof. Let $\{H_1, \dots, H_m\}$ be the set of all facets of Δ . The simplicial complex K is the one associated to Δ and $\tau \in K$ iff $\Delta_{\tau} := \cap_{i \in \tau} H_i \neq \emptyset$. Let F_a be a connected component of F . Then by Lemma 3.4 [19] and our assumption of connected isotropy groups, the stabilizers of every $x \in F_a$ in W coincide. Let W_a be the stabilizer of points in F_a . Let $\mu : \mathcal{Z}_K \rightarrow \Delta$ be the moment map. Note that this is the quotient map by the action of \mathbb{T} . First we show

Lemma 5.18. *$F_a = \mu^{-1}(\Delta_{\sigma})$ for some $\sigma \in K$. In particular, the stabilizer of each point $x \in F_a$ is $W_{\sigma} := W \cap \mathbb{T}_{\sigma}$.*

Proof of Lemma 5.18: For $x \in F_a$, let $\sigma_x := \{i \in [m], x_i = 0\}$. Then $\mu^{-1}(\Delta_{\sigma_x}) \subset F_a$ and the unique stabilizer for F_a is given by $W_a = W \cap \mathbb{T}_{\sigma_x}$. Note that $\sigma_x \neq \emptyset$ since $m > n$. It suffices to show that there is an element $x \in F_a$ such that σ_x is the unique minimal subset among the collection of subsets, $\{\sigma_y \mid y \in F_a\}$. Let σ_x and σ_y be minimal for some $x, y \in F_a$. Suppose that $\sigma_x \neq \sigma_y$ and consider $z \in \mu^{-1}(\Delta_{\sigma_x \cap \sigma_y})$ such that $\sigma_z = \sigma_x \cap \sigma_y$. Since $W_a = W \cap \mathbb{T}_x \cap \mathbb{T}_{\sigma_y} = W_z$, $z \in F$ by dimension counting. The connectivity of F_a then implies that $z \in F_a$. This contradicts to the assumption that σ_x and σ_y are minimal, so $\sigma_x = \sigma_y$. \square

Now let $\{F_{\sigma}\}$ be the set of connected components of F where $F_{\sigma} = \mu^{-1}(\Delta_{\sigma})$. For each σ , choose a splitting $W = W_{\sigma} \times (W/W_{\sigma})$. The target of the injectivity map is computed as follows:

$$H_W(F, \mathbb{Z}) = \bigoplus_{\sigma} H_W(F_{\sigma}, \mathbb{Z}) = \bigoplus_{\sigma} H(BW \times_W F_{\sigma}, \mathbb{Z}) = \bigoplus_{\sigma} H(BW_{\sigma}, \mathbb{Z}) \otimes H(F_{\sigma}/(W/W_{\sigma}), \mathbb{Z}).$$

Now we show that $F_{\sigma}/(W/W_{\sigma})$ is a compact toric symplectic manifold so that $H(F_{\sigma}/(W/W_{\sigma}), \mathbb{Z})$ has only even degree and has no \mathbb{Z} -torsion. Since $[F_{\sigma}/\mathbb{G}]$ is the suborbifold of W/\mathbb{G} -fixed orbifold points, it is a symplectic orbifold (c.f. [23, p.4210, Cor 3.8]), which is compact. Since the unique stabilizer of points of F_{σ} in \mathbb{G} is given by $\mathbb{G}_{\sigma} = \mathbb{G} \cap W_{\sigma}$, $F_{\sigma}/(\mathbb{G}/\mathbb{G}_{\sigma})$ is a compact toric manifold with the effective Hamiltonian action of $(\mathbb{T}/\mathbb{G})/(\mathbb{T}_{\sigma}/\mathbb{G}_{\sigma})$. On the other hand, $F_{\sigma}/(\mathbb{G}/\mathbb{G}_{\sigma})$ is exactly $F_{\sigma}/W = F_{\sigma}/(W/W_{\sigma})$. Thus $H(F_{\sigma}/(W/W_{\sigma}), \mathbb{Z})$ has only even degree and has no \mathbb{Z} -torsion. Now the injectivity of $H_W(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_W(F, \mathbb{Z})$ implies that $H_W(\mathcal{Z}_K, \mathbb{Z})$ has no \mathbb{Z} -torsion and has only even degree. It also implies the freeness of $H_{\mathbb{T}}(\mathcal{Z}_K, \mathbb{Z})$ over $\mathbb{Z}[(\mathbb{T}/W)^*]$ by Theorem 5.5.

To apply the injectivity theorem over \mathbb{Z} (Remark 4.10 [19]), we need to have that W_{σ} is connected and the weights of the action on the (negative) normal bundle are all primitive for each connected component of F_{σ} . The former is true by the assumption. For the latter, look at the normal bundle of F_{σ} in \mathcal{Z}_K which is given by $\bigoplus_{i \in \sigma} \mathbb{C} \frac{\partial}{\partial z_i}$. The weights of the \mathbb{T}_{σ} -equivariant normal bundle are the standard \mathbb{Z} -basis $\{\lambda_i, i \in \sigma\}$ of $N_{\mathbb{T}_{\sigma}}^*$. We need to check that the induced W_{σ} -weights $\tilde{\lambda}_i := A_{\sigma}^*(\lambda_i) \in N_{W_{\sigma}}^*$ are non-zero and primitive where $A_{\sigma} : W_{\sigma} \hookrightarrow T_{\sigma}$ is the restriction of the natural inclusion $A : W \rightarrow \mathbb{T}$. It is easy to see that $\tilde{\lambda}_i$ is non-zero, since, if otherwise, the normal direction $\mathbb{C} \frac{\partial}{\partial z_i}$ is also contained in F_a . Finally the proof is completed by the following lemma.

Lemma 5.19. *$A_{\sigma}^*(\lambda_i) \in N_{W_{\sigma}}^*$ is primitive.*

Proof of Lemma 5.19: Consider the following commutative diagram of tori and its dual for the weight lattices:

$$\begin{array}{ccc}
 W_{\sigma \setminus \{i\}} & \longrightarrow & T_{\sigma \setminus \{i\}} \\
 \downarrow & & \downarrow \\
 W_{\sigma} & \xrightarrow{A_{\sigma}} & T_{\sigma} \\
 & \searrow f_i & \downarrow g_i \\
 & & T_{\{i\}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 N_{W_{\sigma \setminus \{i\}}}^* & \longleftarrow & N_{T_{\sigma \setminus \{i\}}}^* \\
 \uparrow & & \uparrow \\
 N_{W_{\sigma}}^* & \xleftarrow{A_{\sigma}^*} & N_{T_{\sigma}}^* \\
 & \nwarrow f_i^* & \uparrow g_i^* \\
 & & N_{T_{\{i\}}}^*
 \end{array}
 \tag{4}$$

Here g_i is the canonical projection. The map $f_i = g_i \circ A_{\sigma}$ must be surjective since T_i is one dimensional and $\tilde{\lambda}_i$ is non-zero. Also we have $W_{\sigma \setminus \{i\}} = \ker f_i$ which must be connected by the assumption. Therefore we have a short exact sequence $0 \rightarrow W_{\sigma \setminus \{i\}} \rightarrow W_{\sigma} \rightarrow T_{\{i\}} \rightarrow 0$ which implies that f_i^* maps $N_{T_i}^*$ to a direct summand. Thus $\tilde{\lambda}_i$ must be primitive since λ_i is a basis of $N_{T_i}^*$. \square

Remark 5.20. Theorem 5.17 holds when $G = W$. In this case, by the assumption, G acts on \mathcal{Z}_K is free and $F = \mathcal{Z}_K$. We recover the fact that the equivariant cohomology of toric manifolds (or smooth toric varieties) is free over $\mathbb{Z}[\mathbb{R}^*]$.

6. Examples

Example 6.1 (Effective Weighted Projective Spaces). Let $\mathbf{a} := (a_1, \dots, a_m)$ be a sequence of positive integers with $\gcd(a_1, \dots, a_m) \neq 1$ and let $[\mathbb{CP}_{\mathbf{a}}^{m-1}]$ be the corresponding effective weighed projective space. As in Section 2, $H^*([\mathbb{CP}_{\mathbf{a}}^{m-1}]; \mathbb{Z}) = H_{\mathbf{G}}^*(\mathcal{Z}_K; \mathbb{Z})$ where $\mathbf{G} = \{(t^{a_1}, \dots, t^{a_m})\} \subset T$ and K the boundary of an $(m-1)$ -simplex. Here we will not write the matrix B . It would be useful if there is a formula to describe a \mathbb{Z} -basis of the dual weight lattice of \mathbb{R} in terms of (a_1, \dots, a_m) but to our knowledge it is not known.

The corresponding Stanley-Reisner ring is $\mathbb{Z}[x_1, \dots, x_m]/\langle x_1 \cdots x_m \rangle$. In [20], the ordinary cohomology is computed:

$$H_{\mathbf{G}}^*(\mathcal{Z}_K, \mathbb{Z}) \cong \mathbb{Z}[y]/\langle a_1 \cdots a_m y^m \rangle.$$

It has only even degree, so $H_T^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_{\mathbf{G}}^*(\mathcal{Z}_K, \mathbb{Z})$ is surjective. However $H_{\mathbf{G}}^*(\mathcal{Z}_K, \mathbb{Z})$ has \mathbb{Z} -torsion, and so $H_T^*(\mathcal{Z}_K, \mathbb{Z})$ is not free over $\mathbb{Z}[\mathbb{R}^*]$.

Thus $H_T^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_{\mathbf{G}}^*(\mathcal{Z}_K, \mathbb{Z})$ is surjective, but the source is not free over $\mathbb{Z}[\mathbb{R}^*]$. $[\mathbb{CP}_{12}^1]$ is the simplest such example where the dimension is 2. This provides an answer to a question analogous to Question 1.1 [17], see also [2].

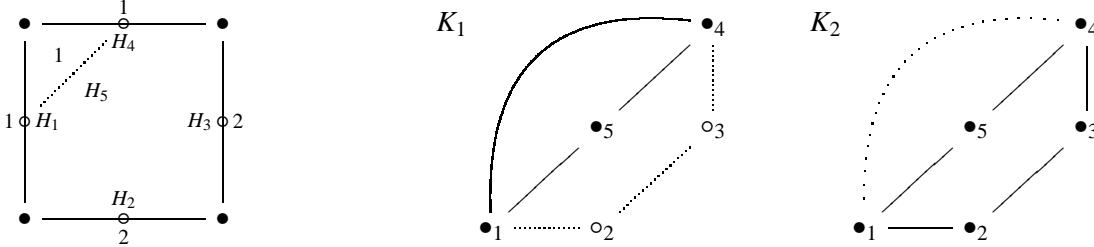
Example 6.2. As we saw in the previous examples, $H_{\mathbf{G}}(\mathcal{Z}_K, \mathbb{Z})$ has \mathbb{Z} -torsion (infinitely many) in even degree for $[\mathcal{Z}_K/\mathbf{G}] = \mathbb{CP}_{\mathbf{a}}^{m-1}$. The direct product of such toric orbifolds is also a toric orbifold and the Künneth theorem provides the \mathbb{Z} -torsions in odd degree. More concretely, take the labeled polytope

$$\begin{array}{c}
 \bullet \quad \frac{1}{H_4} \quad \bullet \\
 \left| \begin{array}{cc} H_1 & H_3 \\ H_2 & \end{array} \right| 2 \\
 \bullet \quad \frac{2}{2} \quad \bullet
 \end{array}$$

which gives $B = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$. This defines the direct product $[\mathbb{CP}_{12}^1 \times \mathbb{CP}_{12}^1]$ which gives odd degree elements in $H_T(\mathcal{Z}_K, \mathbb{Z})$. Thus $H_T^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_{\mathbf{G}}^*(\mathcal{Z}_K, \mathbb{Z})$ is not surjective. We can also see this by checking

if $(x_1 - 2x_3, 2x_2 - x_4)$ is a regular sequence of $\mathbb{Z}[K] = \frac{\mathbb{Z}[x_1, \dots, x_4]}{\langle x_1 x_3, x_2 x_4 \rangle}$ as a module over $\mathbb{Z}[x_1 - 2x_3, 2x_2 - x_4]$. Indeed, it is not a regular sequence: $x_1 - 2x_3$ is a non-zero divisor in $\mathbb{Z}[K]$ but $2x_2 - x_4$ is a zero divisor in $\mathbb{Z}[K]/(x_1 - 2x_3)$ since $(2x_2 - x_4)x_2x_3^2 = 0$ and $x_2x_3^2 \neq 0$ in $\mathbb{Z}[K]/(x_1 - 2x_3)$.

From this example, we can create more examples by the method of the symplectic cut. Consider the cutting by a hyperplane H_5 :



where K_1 and K_2 are simplicial complexes associated to the cut pieces. Let $\tilde{B} = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & -1 & -1 \end{pmatrix}$. Then \tilde{B} defines a 2-dimensional subtorus G of 5 dimensional torus T , which acts on \mathcal{Z}_{K_1} and \mathcal{Z}_{K_2} locally freely. Each $[\mathcal{Z}_{K_1}/G]$ and $[\mathcal{Z}_{K_2}/G]$ defines the toric orbifolds corresponding to the symplectic cut of $\mathbb{CP}_{12}^1 \times \mathbb{CP}_{12}^1$. The surjectivity holds for K_1 but not for K_2 . We can see this by checking that, if $u_1 = x_1 - 2x_3 + x_4$ and $u_2 = 2x_1 - x_4 - x_5$, then (u_1, u_2) is a regular sequence of $\mathbb{Z}[K_1]$, while (u_1, u_2) is not a regular sequence for $\mathbb{Z}[K_2]$ since $2x_1 - x_4 - x_5$ is annihilated by $x_2x_3^2$. By the same algebraic computation, we see that (u_1, u_2) is not a regular sequence of $\mathbb{Z}[K_1 \cup K_2]$ as a $\mathbb{Z}[u_1, u_2]$ -module, i.e. $H_T(\mathcal{Z}_{K_1 \cup K_2}, \mathbb{Z}) \rightarrow H_G(\mathcal{Z}_{K_1 \cup K_2}, \mathbb{Z})$ is not surjective.

7. Algebraic Gysin Sequence

Let U be a subgroup of G such that $L := G/U$ is a 1-dimensional torus. We have a principal L -bundle $\pi : ET \times_U \mathcal{Z}_K \rightarrow ET \times_G \mathcal{Z}_K$ and the corresponding Gysin sequence

$$\cdots \rightarrow H_G^{i-1}(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\cup e} H_G^{i+1}(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\pi^*} H_U^{i+1}(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\pi_*} H_G^i(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\cup e} H_G^{i+2}(\mathcal{Z}_K, \mathbb{Z}) \rightarrow \cdots$$

where e is the Euler class of the bundle and π_* / π^* is the pushforward / pullback map. In the light of Theorem 3.3, it is natural to ask if there is a purely algebraic construction of a long exact sequence of Tor's corresponding the Gysin sequence. We describe the construction in the following

Construction 7.1 (Algebraic Gysin Sequence). Let $\tilde{R} := T/U$ and identify $H^*(BR, \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n, u_{n+1}] = \mathbb{Z}[\tilde{R}^*]$ where $\mathbb{Z}[u_1, \dots, u_n] = H^*(BR, \mathbb{Z})$. Consider the short exact sequence of Koszul complexes for (as modules over $\mathbb{Z}[\tilde{R}^*]$):

$$0 \rightarrow K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \xrightarrow{\tau^*} K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n, \xi_{n+1}) \xrightarrow{\tau_*} K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n)[-1] \rightarrow 0 \quad (5)$$

where the first map is the obvious inclusion denoted by τ^* and the second map is *getting rid of* $\xi_{n+1} \wedge$ denoted by τ_* . Note that $K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) = \mathbb{Z}[u_1, \dots, u_n, u_{n+1}]\langle \xi_1, \dots, \xi_n \rangle$ and the differential is giving by extending $\partial \xi_i = u_i$ as a differential algebra where $\langle \rangle$ denotes the exterior algebra.

Let M be a graded $\mathbb{Z}[x_1, \dots, x_m]$ -module. After tensoring M , we obtain the long exact sequence of Tor modules over $\mathbb{Z}[u_1, \dots, u_{n+1}]$:

$$\cdots \rightarrow \text{Tor}_{i+1}^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\delta} \text{Tor}_{i+1}^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\tau^*} \text{Tor}_{i+1}^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\tau_*} \text{Tor}_i^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\delta} \text{Tor}_i^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \rightarrow \cdots \quad (6)$$

We call this the *algebraic (homological) Gysin sequence*.

Proposition 7.2. *The connecting map δ is a multiplication by u_{n+1} and it is independent of the choice of u_{n+1} .*

Proof. It follows from the diagram chasing. Consider the part of the map of complexes

$$\begin{array}{ccccc} K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \\ \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\ K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \end{array}$$

Let z be a cycle in $K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n)$ and lift it to top left corner:

$$\begin{array}{ccccc} u_i \cdot z & \longrightarrow & \partial(\xi_{n+1} \wedge z) = u_i \cdot z & \longrightarrow & 0 \\ & & \uparrow \partial & & \uparrow \partial \\ & & \xi_{n+1} \wedge z & \longrightarrow & z \end{array}$$

Since Tor modules are independent of the choice of the basis of the polynomial ring, so δ is independent of the choice of u_{n+1} . \square

Definition 7.3 (Cohomological Algebraic Gysin sequence). As in Remark 2.2, we can assign the cohomological degree and turn the sequence (6) into a cohomological sequence:

$$\cdots \rightarrow \mathrm{Tor}_{\mathbb{Z}[\tilde{R}^*]}^{i-1}(\mathbb{M}, \mathbb{Z}) \xrightarrow{\cdot u_{n+1}} \mathrm{Tor}_{\mathbb{Z}[\tilde{R}^*]}^{i+1}(\mathbb{M}, \mathbb{Z}) \xrightarrow{\tau^*} \mathrm{Tor}_{\mathbb{Z}[\tilde{R}^*]}^{i+1}(\mathbb{M}, \mathbb{Z}) \xrightarrow{\tau_*} \mathrm{Tor}_{\mathbb{Z}[\tilde{R}^*]}^i(\mathbb{M}, \mathbb{Z}) \xrightarrow{\cdot u_{n+1}} \mathrm{Tor}_{\mathbb{Z}[\tilde{R}^*]}^{i+2}(\mathbb{M}, \mathbb{Z}) \rightarrow \cdots \quad (7)$$

We call this the *cohomological algebraic Gysin sequence*.

Remark 7.4. In the special case when K is the simplicial complex associated to a Delzant polytope Δ , \mathcal{Z}_K/S is homeomorphic to the corresponding symplectic toric manifold. We have $\mathrm{Tor}_i^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ for all $i \geq 1$ since we know that $\mathbb{Z}[K]$ is free over $\mathbb{Z}[\tilde{R}^*]$. Thus the long exact sequence (7.1) implies that $\mathrm{Tor}_i^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ for all $i \geq 2$. Hence the only non-zero part of the long exact sequence is:

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \xrightarrow{\tau_*} \mathrm{Tor}_0^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \xrightarrow{\delta} \mathrm{Tor}_0^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \xrightarrow{\tau^*} \mathrm{Tor}_0^{\mathbb{Z}[\tilde{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \longrightarrow 0$$

This sequence, together with the identification of torsion algebras and cohomology rings of toric manifolds, gives the Gysin sequence used in Luo's paper [24] to compute the cohomology ring of a good contact toric manifold.

ACKNOWLEDGEMENTS

The authors want to thank M. Franz, T. Holm, Y. Karshon, A. Knutson, T. Ohmoto, K. Ono, D. Suh for important advice and useful conversations. The first author is particularly indebted to K. Ono for providing him an excellent environment at Hokkaido University where he had spent significant time for this paper in July and August 2011. The first author would like to show his gratitude to the Algebraic Structure and its Application Research Center (ASARC) at KAIST for its constant support. The first author is also supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2012-0000795, 2011-0001181).

REFERENCES

- [1] ADEM, A., LEIDA, J., AND RUAN, Y. *Orbifolds and stringy topology*, vol. 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.
- [2] ALLDAY, C. A family of unusual torus group actions. In *Group actions on manifolds (Boulder, Colo., 1983)*, vol. 36 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1985, pp. 107–111.
- [3] BASKAKOV, I. V., BUKHSHTABER, V. M., AND PANOV, T. E. Algebras of cellular cochains, and torus actions. *Uspekhi Mat. Nauk* 59, 3(357) (2004), 159–160.
- [4] BORISOV, L. A., CHEN, L., AND SMITH, G. G. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.* 18, 1 (2005), 193–215 (electronic).
- [5] BRUNS, W., AND HERZOG, J. *Cohen-Macaulay rings*, vol. 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.
- [6] BUCHSTABER, V. M., AND PANOV, T. E. *Torus actions and their applications in topology and combinatorics*, vol. 24 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.
- [7] BUKHSHTABER, V. M., AND PANOV, T. E. Torus actions and the combinatorics of polytopes. *Tr. Mat. Inst. Steklova* 225, Solitony Geom. Topol. na Perekrest. (1999), 96–131.
- [8] COX, D. A. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.* 4, 1 (1995), 17–50.
- [9] DANILOV, V. I. The geometry of toric varieties. *Uspekhi Mat. Nauk* 33, 2(200) (1978), 85–134, 247.
- [10] DAVIS, M. W., AND JANUSZKIEWICZ, T. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.* 62, 2 (1991), 417–451.
- [11] EDIDIN, D. Equivariant geometry and the cohomology of the moduli space of curves. [arXiv:1006.2364](#).
- [12] EDIDIN, D., AND GRAHAM, W. Equivariant intersection theory. *Invent. Math.* 131, 3 (1998), 595–634.
- [13] FRANZ, M. Koszul duality and equivariant cohomology for tori. *Int. Math. Res. Not.*, 42 (2003), 2255–2303.
- [14] FRANZ, M. The integral cohomology of toric manifolds. *Tr. Mat. Inst. Steklova* 252, Geom. Topol., Diskret. Geom. i Teor. Mnoz. (2006), 61–70.
- [15] FRANZ, M. Describing toric varieties and their equivariant cohomology. *Colloq. Math.* 121, 1 (2010), 1–16.
- [16] FRANZ, M., AND PUPPE, V. Exact cohomology sequences with integral coefficients for torus actions. *Transform. Groups* 12, 1 (2007), 65–76.
- [17] FRANZ, M., AND PUPPE, V. Freeness of equivariant cohomology and mutants of compactified representations. In *Toric topology*, vol. 460 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2008, pp. 87–98.
- [18] HATCHER, A. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [19] HOLM, T., AND MATSUMURA, T. Equivariant cohomology for hamiltonian torus actions on symplectic orbifolds. To Appear in *Transformation Groups*, [arXiv:1008.3315](#).
- [20] HOLM, T. S. Orbifold cohomology of abelian symplectic reductions and the case of weighted projective spaces. In *Poisson geometry in mathematics and physics*, vol. 450 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2008, pp. 127–146.
- [21] JURKIEWICZ, J. Torus embeddings, polyhedra, k^* -actions and homology. *Dissertationes Math. (Rozprawy Mat.)* 236 (1985), 64.
- [22] LERMAN, E., AND MALKIN, A. Hamiltonian group actions on symplectic Deligne-Mumford stacks and toric orbifolds. *Adv. Math.* 229, 2 (2012), 984–1000.
- [23] LERMAN, E., AND TOLMAN, S. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.* 349, 10 (1997), 4201–4230.
- [24] LUO, S. Cohomology rings of good contact toric manifolds. [arXiv:1012.2146](#).
- [25] MATSUMURA, H. *Commutative algebra*, second ed., vol. 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.
- [26] MCCLEARY, J. *A user's guide to spectral sequences*, second ed., vol. 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001.
- [27] PODDAR, M., AND SARKAR, S. On quasitoric orbifolds. *Osaka J. Math.* 47, 4 (2010), 1055–1076.
- [28] ROMAGNY, M. Group actions on stacks and applications. *Michigan Math. J.* 53, 1 (2005), 209–236.
- [29] SERRE, J.-P. *Local algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author.
- [30] TOLMAN, S. *Group Actions And Cohomology*. PhD dissertation, Harvard University, Dept. Math., 1993.

CORNELL UNIVERSITY, MATHEMATICS DEPARTMENT, ITHACA, NY 14853

E-mail address: sl943@cornell.edu

ALGEBRAIC STRUCTURE AND ITS APPLICATIONS RESEARCH CENTER, DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, SOUTH KOREA

E-mail address: tooomatsumura@kaist.ac.kr

DEPARTMENT OF MATHEMATICS, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NC 27109

E-mail address: moorewf@wfu.edu